PHYS 331 - September 11, 2023

Lock-in Detection

Lock-in detectors are good at meas. the amplitude & phase of small AC signals when the freq. of the desired signal is known.

Recall that we can represent any complex number using:

① mag. & phase

$$z = |z| e^{j\phi} = |z| \overbrace{(\cos\phi + j\sin\phi)}^{e^{j\phi}}$$

② real & imagary component

$$z = \underset{\text{real}}{x} + j\underbrace{y}_{\text{imaginary}}$$

Switch between the two representations:

$$|z| = \sqrt{x^2 + y^2}$$

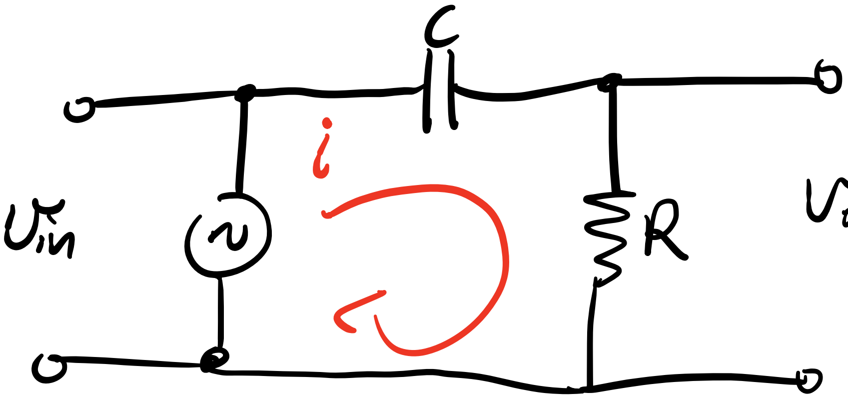
$$\tan \phi = \frac{y}{x}$$

$$x = |z| \cos \phi$$

$$y = |z| \sin \phi$$

We want to show that a lock-in detector can be used to find the real (x) and imaginary (y) components of a signal. Then we can calc. the signal amp. from $\sqrt{x^2+y^2}$ & signal phase from $\phi = \tan^{-1}\left(\frac{y}{x}\right)$.

To understand the physical meaning of the real & imaginary components of a signal, let's consider an RC circuit.



$$v_{out} = V_{out} e^{j(wt + \phi)}$$
$$= iR$$

$$v_{in} = V_{in} e^{jwt}$$

$$i = \frac{v_{in}}{Z} \qquad Z = R + \underbrace{\frac{1}{jwC}}_{Z_C} = \frac{jwCR + 1}{jwC}$$

$$i = v_{in}\left(\frac{jwC}{1 + jwRC}\right)\frac{(1 - jwRC)}{(1 - jwRC)}$$

$$= v_{in}\,\frac{wC(j + wRC)}{1 + (wRC)^2}$$

$$\underline{v_{out}} = iR = v_{in}\,\frac{wRC(j + wRC)}{1 + (wRC)^2}$$

sub in $v_{in} = V_{in} e^{j\omega t}$

$v_{out} = V_{out} e^{j(\omega t + \phi)} = V_{out} e^{j\omega t} e^{j\phi}$

$$V_{out} \cancel{e^{j\omega t}} \underbrace{e^{j\phi}}_{\cos\phi + j\sin\phi} = V_{in} \cancel{e^{j\omega t}} \frac{\omega RC(j + \omega RC)}{1 + (\omega RC)^2}$$

$$V_{out}(\cos\phi + j\sin\phi) = V_{in} \frac{\omega RC(j + \omega RC)}{1 + (\omega RC)^2}$$

set real & imaginary parts equal on left & right sides.

Real:

$$V_{out}\cos\phi = V_{in} \frac{(\omega RC)^2}{1 + (\omega RC)^2} \equiv X$$

Imaginary: $$V_{out}\sin\phi = \frac{V_{in}\,\omega RC}{1 + (\omega RC)^2} \equiv Y$$

X is the "real" component of V_{out}

y is the "imaginary" component of V_{out}.

If we know x & y, we can calc. the amp. & phase of V_{out}.

$$\boxed{\sqrt{x^2+y^2} = V_{out} = \sqrt{\frac{V_{in}^2\left[(wRC)^2+(wRC)^4\right]}{\left[1+(wRC)^2\right]^2}}}$$

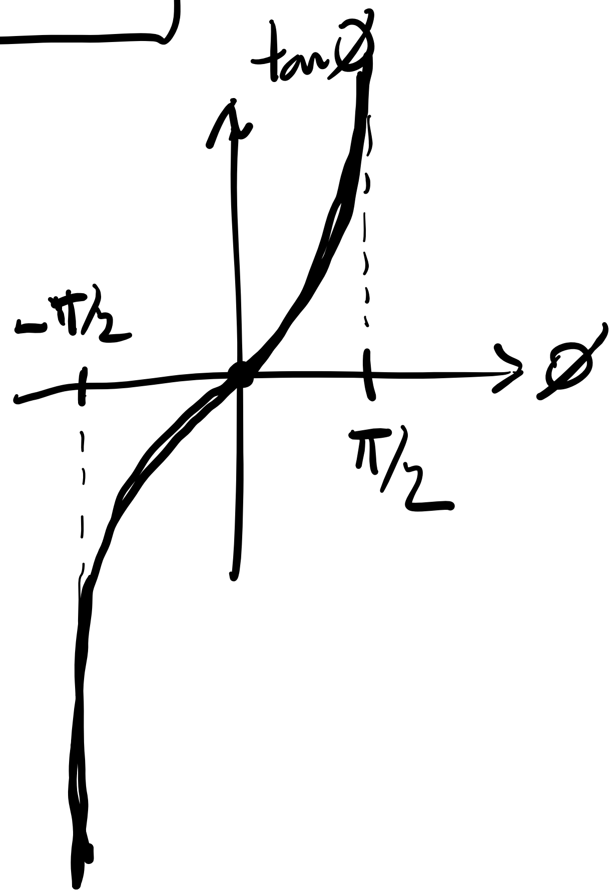
$$= \frac{V_{in}\, wRC\sqrt{1+(wRC)^2}}{1+(wRC)^2}$$

$$\therefore \boxed{V_{out} = V_{in}\frac{wRC}{\sqrt{1+(wRC)^2}}}$$ high-pass filter.

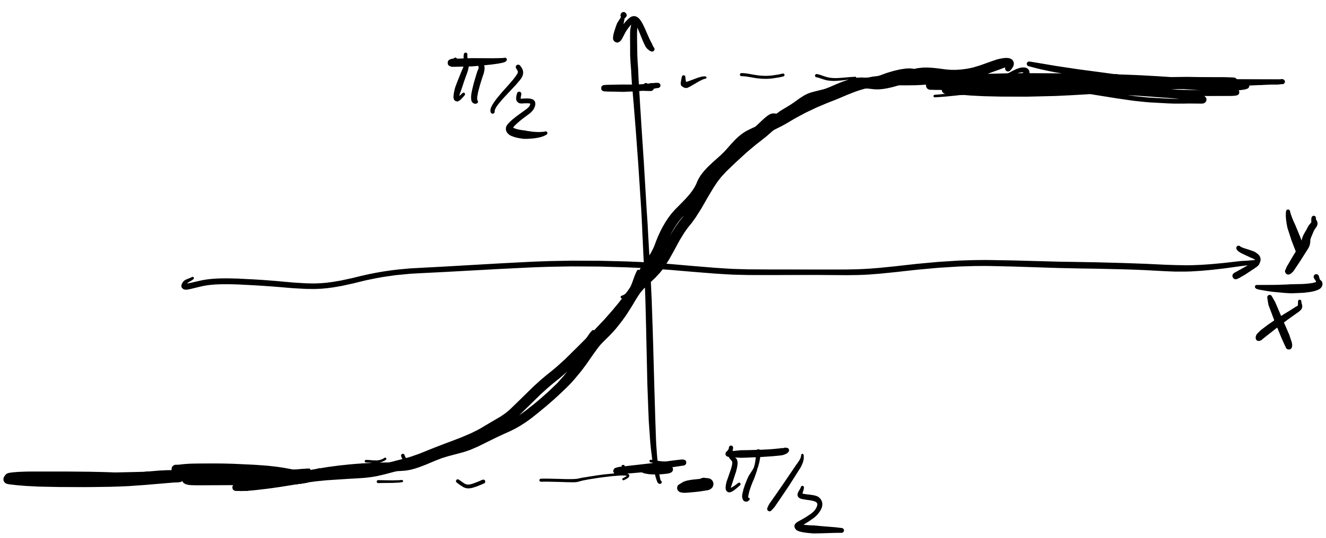
Phase: $$\tan\varnothing = \frac{y}{x} = \frac{\frac{\cancel{V_{in}}\,\cancel{wRC}}{\cancel{1+(wRC)^2}}}{\frac{\cancel{V_{in}}\,(wRC)^{\cancel{2}}}{\cancel{1+(wRC)^2}}}$$

$$\boxed{\tan\phi = \frac{1}{\omega RC}}$$

Interpret ϕ.



$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$



Notice if the imaginary component of a signal is zero ($y=0$), then

$$\frac{y}{x}=0 \; \& \; \boxed{\phi=0.}$$

When an output is shifted in phase relative to an input signal, it implies that the output has both real & imaginary components.